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AUTHOR(S):

CHA, JAE CHOON; FRIEDL, STEFAN; KIM, TAEHEE

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# COBORDISM ON HOMOLOGY CYLINDERS AND COMBINATORIAL TORSIONS

JAE CHOON CHA, STEFAN FRIEDL, AND TAEHEE KIM

## 1. INTRODUCTION

In this extended abstract, we survey the results in [CFK09] of the authors. This extended abstract contains no original results. Let  $\Sigma_{g,n}$  be an oriented compact surface of genus  $g$  with  $n$  boundary components. The *mapping class group*  $\mathcal{M}_{g,n}$  is the group of orientation-preserving isotopy classes of automorphisms of  $\Sigma_{g,n}$  which reduce to the identity on  $\partial\Sigma_{g,n}$ . The mapping class group has been one of the central research subjects in many mathematical areas, and we refer the reader to [FM11] for more details about the mapping class group.

Recently an enlargement of the mapping class group has been introduced. Goussarov [Go99] and Habiro [Ha00] introduced the notion of *homology cylinder* and Garoufalidis and Levine [GL05][Le01] introduced the homology cobordism group of homology cylinders. Roughly speaking, a homology cylinder over  $\Sigma_{g,n}$  is a cobordism between surfaces equipped with markings (diffeomorphisms) to  $\Sigma_{g,n}$  where the cobordism is required to be homologically a product. The isotopy classes of homology cylinders over  $\Sigma_{g,n}$  form a monoid under juxtaposition and this monoid is denoted by  $\mathcal{C}_{g,n}$ . By considering categorical differences, we obtain a group  $\mathcal{H}_{g,n}^{\text{smooth}}$  (resp.  $\mathcal{H}_{g,n}^{\text{top}}$ ) of smooth (resp. topological) homology cobordism classes of homology cylinders. Henceforth, we will use the notation  $\mathcal{H}_{g,n}$  for both  $\mathcal{H}_{g,n}^{\text{smooth}}$  and  $\mathcal{H}_{g,n}^{\text{top}}$  when the concerned results hold in both categories. See Section 2 for the precise definitions of homology cylinders and their cobordism groups.

We say that  $\mathcal{C}_{g,n}$  and  $\mathcal{H}_{g,n}$  are enlargements of  $\mathcal{M}_{g,n}$  since  $\mathcal{M}_{g,n}$  injects into  $\mathcal{C}_{g,n}$  and  $\mathcal{H}_{g,n}$ . More precisely, there is a map  $\mathcal{M}_{g,n} \rightarrow \mathcal{C}_{g,n} \rightarrow \mathcal{H}_{g,n}$  which is injective [CFK09, Proposition 2.4]. (Also see [GL05, Section 2.4] and [Le01, Section 2.1] for the case  $n = 1$ .) Therefore some natural questions arise regarding the comparison between the structures of  $\mathcal{M}_{g,n}$  and  $\mathcal{H}_{g,n}$ . For instance, it is known that the mapping class group is finitely presented [BH71, Mc75] and perfect if  $g \geq 3$  [Po78]. Regarding the homology cylinder, Goda and Sakasai [GS09] ask if  $\mathcal{H}_{g,1}^{\text{smooth}}$  is a perfect group and Garoufalidis and Levine [GL05] ask if  $\mathcal{H}_{g,1}^{\text{smooth}}$  is infinitely generated. Finally in [CFK09] the authors showed that if  $b_1(\Sigma_{g,n}) > 0$  then  $\mathcal{H}_{g,n}$  is not a perfect group and not finitely generated, answering the questions of Goda-Sakasai and Garoufalidis-Levine.

**Theorem 1.1.** [CFK09, Theorem 1.2 and Theorem 1.3]

- (1) If  $b_1(\Sigma_{g,n}) > 0$ , then there exists an epimorphism

$$\mathcal{H}_{g,n} \rightarrow (\mathbb{Z}/2)^\infty$$

which splits (i.e., there is a right inverse). In particular, the abelianization of  $\mathcal{H}_{g,n}$  contains a direct summand isomorphic to  $(\mathbb{Z}/2)^\infty$ .

(2) If  $n > 1$ , then there exists an epimorphism

$$\mathcal{H}_{g,n} \rightarrow \mathbb{Z}^\infty.$$

Furthermore, the abelianization of  $\mathcal{H}_{g,n}$  contains a direct summand isomorphic to  $(\mathbb{Z}/2)^\infty \oplus \mathbb{Z}^\infty$ .

In particular, this shows that the structure of  $\mathcal{H}_{g,n}$  is much different from that of  $\mathcal{M}_{g,n}$ .

In this article, we will survey the proof of Theorem 1.1(1) in [CFK09] and the related materials such as the *torsion* invariant (see Section 2). The reader is referred to [CFK09] for more results such as the proof of Theorem 1.1(2), the difference between  $\mathcal{H}_{g,n}^{\text{top}}$  and  $\mathcal{H}_{g,n}^{\text{smooth}}$ , and the Torelli group analogue for  $\mathcal{H}_{g,n}$ . We note that the proof of Theorem 1.1(2) also uses the torsion invariant but the argument is more elaborate than in the proof of Theorem 1.1(1).

## 2. HOMOLOGY CYLINDERS AND TORSION INVARIANTS

### 2.1. Homology cylinders.

**Definition 2.1.** (1) A *homology cylinder*  $(M, i_+, i_-)$  over a compact surface  $\Sigma$  is defined to be a 3-manifold  $M$  together with injections (markings)  $i_+, i_-: \Sigma \rightarrow \partial M$  satisfying the following:

- (a)  $i_+$  is orientation preserving and  $i_-$  is orientation reversing.
  - (b)  $\partial M = i_+(\Sigma) \cup i_-(\Sigma)$  and  $i_+(\Sigma) \cap i_-(\Sigma) = i_+(\partial\Sigma) = i_-(\partial\Sigma)$ .
  - (c)  $i_+|_{\partial\Sigma} = i_-|_{\partial\Sigma}$ .
  - (d)  $i_+, i_-: H_*(\Sigma) \rightarrow H_*(M)$  are isomorphisms.
- (2) Two homology cylinders  $(M, i_+, i_-)$  and  $(N, j_+, j_-)$  over  $\Sigma = \Sigma_{g,n}$  are called *isomorphic* if there exists an orientation-preserving diffeomorphism  $f: M \rightarrow N$  satisfying  $j_\pm = f \circ i_\pm$ .

An example of homology cylinder is given using a mapping class  $\varphi \in \mathcal{M}_{g,n}$  as follows: Let  $M(\varphi) = (\Sigma_{g,n} \times [0, 1] / \sim, i_+ = \text{id} \times 0, i_- = \varphi \times 1)$  where  $\sim$  is given by  $(x, s) \sim (x, t)$  for  $x \in \partial\Sigma_{g,n}$  and  $s, t \in [0, 1]$ . Then  $M(\varphi)$  is a homology cylinder. In particular, when  $\varphi = \text{id}$ , we call the resulting homology cylinder the *product homology cylinder*. The isotopy classes of homology cylinders over  $\Sigma_{g,n}$  form a monoid under juxtaposition:  $(M, i_+, i_-) \cdot (N, j_+, j_-) := (M \cup_{i_- \circ (j_+)^{-1}} N, i_+, j_-)$ . We denote by  $\mathcal{C}_{g,n}$  the resulting monoid. Note that the data from the markings  $i_-$  and  $j_+$  is used in the definition of the monoid operation.

**Definition 2.2.** Let  $(M, i_+, i_-)$  and  $(N, j_+, j_-)$  be homology cylinders over  $\Sigma_{g,n}$ . Then  $(M, i_+, i_-)$  and  $(N, j_+, j_-)$  are called *smoothly homology cobordant* (resp. topologically homology cobordant) if there exists a compact oriented smooth 4-manifold (resp. topological 4-manifold)  $W$  such that

$$\partial W = M \cup (-N) / i_+(x) = j_+(x), i_-(x) = j_-(x) \quad (x \in \Sigma),$$

and such that the inclusion induced maps  $H_*(M) \rightarrow H_*(W)$  and  $H_*(N) \rightarrow H_*(W)$  are isomorphisms.

The smooth (resp. topological) homology cobordism classes form a group under juxtaposition and we denote the resulting group by  $\mathcal{H}_{g,n}^{\text{smooth}}$  (resp.  $\mathcal{H}_{g,n}^{\text{top}}$ ). In particular, the monoid structure of  $\mathcal{C}_{g,n}$  descends to a group structure of  $\mathcal{H}_{g,n}$  and we have the surjection

$\mathcal{C}_{g,n} \rightarrow \mathcal{H}_{g,n}^{\text{smooth}} \rightarrow \mathcal{H}_{g,n}^{\text{top}}$ . In  $\mathcal{H}_{g,n}$ , the identity is the class of the product homology cylinder and the inverse of  $(M, i_+, i_-)$  is  $(-M, i_-, i_+)$ . In [CFK09, Theorem 1.1], the authors showed that the kernel of the epimorphism  $\mathcal{H}_{0,n}^{\text{smooth}} \rightarrow \mathcal{H}_{0,n}^{\text{top}}$  maps onto an abelian group of infinite rank.

Since  $M(\varphi) \cdot M(\psi) = M(\varphi \circ \psi)$  for  $\varphi, \psi \in \mathcal{M}_{g,n}$ , we have a monoid morphism  $\mathcal{M}_{g,n} \rightarrow \mathcal{C}_{g,n}$  which sends  $\varphi \mapsto M(\varphi)$ . Furthermore, the composition  $\mathcal{M}_{g,n} \rightarrow \mathcal{C}_{g,n} \rightarrow \mathcal{H}_{g,n}$  is injective [CFK09, Proposition 2.4]. (Also see [GL05, Section 2.4] and [Le01, Section 2.1] for the case  $n = 1$ .)

**2.2. The torsion invariant of homology cylinders.** Let  $(M, N)$  be a manifold pair such that  $H_*(M, N; \mathbb{Z}) = 0$  and  $\varphi: \pi_1(M) \rightarrow H$  be an epimorphism to a free abelian group. Let  $Q(H)$  be the quotient field of the group ring  $\mathbb{Z}[H]$ . Let  $p: \tilde{M} \rightarrow M$  be the universal covering map of  $M$  and  $\tilde{N} := p^{-1}(N)$ . Then the *torsion invariant*  $\tau(M, N; Q(H))$  is defined using the chain complex  $C_*(\tilde{M}, \tilde{N}) \otimes_{\mathbb{Z}[\pi_1(M)]} Q(H)$ . (See [Mi66] and [Tu01] for the definition of the torsion.)

Let  $(M, i_+, i_-)$  be a homology cylinder over  $\Sigma_{g,n}$ . Let  $\Sigma_{\pm} := i_{\pm}(\Sigma)$  in  $M$  and  $H := H_1(\Sigma; \mathbb{Z})$ . We define a homomorphism  $\varphi = \varphi(M) = \varphi((M, i_+, i_-))$  as follows:

$$\varphi: \pi_1(M) \rightarrow H_1(M; \mathbb{Z}) \xleftarrow{\cong} H_1(\Sigma_+; \mathbb{Z}) \xleftarrow{i_+^*} H = H_1(\Sigma; \mathbb{Z}).$$

Since  $H_1(M, \Sigma_+; \mathbb{Z}) = 0$ , we have  $H_1(M, \Sigma_+; Q(H)) = 0$  (see [CFK09, Lemma 3.1]). Now we define the *torsion* of  $(M, i_+, i_-)$  as follows:

**Definition 2.3.** For the homomorphism  $\varphi: \pi_1(M) \rightarrow H_1(M; \mathbb{Z}) \xleftarrow{\cong} H_1(\Sigma_+; \mathbb{Z}) \xleftarrow{i_+^*} H = H_1(\Sigma; \mathbb{Z})$ , we define the *torsion* of  $(M, i_+, i_-)$  to be

$$\tau(M) := \tau(M, \Sigma_+; Q(H)) \in Q(H)^{\times}.$$

The torsion of a homology cylinder is well-defined up to multiplication by  $\pm h$  ( $h \in H$ ) and was first studied by Sakasai [Sa06]. In fact, the torsion of a homology cylinder is computed easily as below. Note that  $\mathbb{Z}[H]$  is a unique factorization domain, and hence for any finitely generated module over  $\mathbb{Z}[H]$ , we can define its *order*, which is an element of  $\mathbb{Z}[H]$ .

**Lemma 2.4.** [CFK09, Lemma 3.2] *For a homology cylinder  $(M, i_+, i_-)$ , the torsion  $\tau(M)$  is the order of  $H_1(M, \Sigma_+; \mathbb{Z}[H])$  as a  $\mathbb{Z}[H]$ -module.*

### 3. A HOMOMORPHISM TO AN ABELIAN GROUP

**3.1. Construction of a homomorphism.** To show Theorem 1.1, we will construct a homomorphism from  $\mathcal{H}_{g,n}$  to an abelian group whose image is isomorphic to  $(\mathbb{Z}/2)^{\infty}$ .

Abusing the notation, for a homology cylinder  $(M, i_+, i_-)$  over  $\Sigma_{g,n}$  and  $H = H_1(\Sigma_{g,n}; \mathbb{Z})$ , we denote the automorphism

$$(i_+)_*^{-1}(i_-)_*: H \xrightarrow{(i_-)_*} H_1(M; \mathbb{Z}) \xrightarrow{(i_+)_*^{-1}} H$$

by  $\varphi(M)$ . And for  $H_{\partial}$ , the image of  $H_1(\partial\Sigma_{g,n}; \mathbb{Z})$  in  $H$ , we define

$$\text{Aut}^*(H) = \{\varphi \in \text{Aut}(H) \mid \varphi \text{ fixes } H_{\partial} \text{ and preserves the intersection form of } \Sigma\}.$$

By [GS08, Proposition 2.3 and Remark 2.4], it is known that  $\varphi(M) \in \text{Aut}^*(H)$ . Furthermore,  $(i_+)_*^{-1}(i_-)_*$  induces an automorphism of  $\mathbb{Z}[H]$  and we also denote it by  $\varphi(M)$ . For  $a, b \in \mathbb{Z}[H]$ , we write  $a \doteq b$  if  $a$  and  $b$  differ by a unit in  $\mathbb{Z}[H]$ .

The proposition below shows that the torsion invariant induces not a homomorphism, but a *crossed* homomorphism on  $\mathcal{C}_{g,n}$ .

**Proposition 3.1.** [CFK09, Proposition 3.5] *Let  $(M, i_+, i_-)$  and  $(N, j_+, j_-)$  be homology cylinders over  $\Sigma_{g,n}$ . Then*

$$\tau(M \cdot N) \doteq \tau(M) \cdot \varphi(M)(\tau(N)).$$

Moreover, the torsion invariant is not a homology cobordism invariant under this setting. That is, the map  $\tau: \mathcal{C}_{g,n} \rightarrow \mathbb{Z}[H]/\pm H$  does not factor through  $\mathcal{H}_{g,n}$ . For the group  $H$ , we equip  $\mathbb{Z}[H]$  with the standard involution taking  $g \mapsto g^{-1}$  for  $g \in H$  and extend it to  $Q(H)$  by setting  $\overline{p \cdot q} = \bar{p} \cdot \bar{q}^{-1}$ .

**Theorem 3.2.** [CFK09, Theorem 3.10] *Let  $M = (M, i_+, i_-)$  and  $N = (N, j_+, j_-)$  be homology cylinders over  $\Sigma_{g,n}$  which are homology cobordant. Then  $\tau(M)$  and  $\tau(N)$  differ by a norm in  $Q(H)$ :*

$$\tau(M) \doteq \tau(N) \cdot q \cdot \bar{q} \in Q(H)^\times$$

for some  $q \in Q(H)^\times$ .

From Proposition 3.1 and Theorem 3.2, it seems that the torsion invariant does not work for our purpose, which is to construct a homomorphism on  $\mathcal{H}_{g,n}$  with abelian image. From now on we will take an appropriate quotient group of  $Q(H)^\times$  to make the torsion give a homomorphism on  $\mathcal{H}_{g,n}$  with abelian image.

Define the subgroup  $A = A(H)$  of  $Q(H)^\times$  to be the subgroup of  $Q(H)^\times$  generated by the set  $\{\pm h \cdot p^{-1} \cdot \varphi(p) \mid h \in H, p \in Q(H)^\times, \text{ and } \varphi \in \text{Aut}^*(H)\}$ , and the subgroup  $N = N(H)$  to be the subgroup  $N(H) = \{\pm h \cdot q \cdot \bar{q} \mid q \in Q(H)^\times, h \in H\}$ . From Proposition 3.1 and Theorem 3.2. we obtain the following theorem:

**Theorem 3.3.** [CFK09, Corollary 3.12] *The torsion invariant gives rise to a group homomorphism*

$$\tau: \mathcal{H}_{g,n} \rightarrow Q(H)^\times / AN,$$

where  $H = H_1(\Sigma_{g,n}; \mathbb{Z})$ .

Since for a homology cylinder  $(M, i_+, i_-)$  the torsion invariant is defined by using only the base manifold  $M$ , one can easily see that the above homomorphism descends to a homomorphism of the quotient of  $\mathcal{H}_{g,n}$  modulo the normal subgroup  $\langle \mathcal{M}_{g,n} \rangle$  generated by the mapping class group  $\mathcal{M}_{g,n}$ .

**3.2. Proof of Theorem 1.1.** In this subsection we give a proof of Theorem 1.1. We define

$$Q(H)^{\text{sym}} = \{p \in Q(H)^\times \mid p = \bar{p} \text{ in } Q(H)^\times / A\}.$$

Then  $AN \subset Q(H)^{\text{sym}}$ . For  $p, q \in \mathbb{Z}[H]^\times$ , define  $p \sim q$  if  $p \doteq \varphi(q)$  for some  $\varphi \in \text{Aut}^*(H)$ . One easily sees that this gives an equivalence relation on  $\mathbb{Z}[H]^\times$ . Then we say that  $p \in \mathbb{Z}[H]^\times$  is *self-dual* if  $p \sim \bar{p}$ . For each irreducible element  $\lambda \in \mathbb{Z}[H]$ , define  $e_\lambda: Q(H)^\times \rightarrow \mathbb{Z}$  where for  $p \in Q(H)$ ,  $e_\lambda(p)$  is the sum of exponents of distinct irreducible factors  $\mu$  of  $p$  such that  $\mu \sim \lambda$ . Now we have the following proposition.

**Proposition 3.4.** [CFK09, Proposition 5.1] *For a self-dual irreducible element  $\lambda \in \mathbb{Z}[H]$ , the map*

$$\Psi_\lambda: Q(H)^{\text{sym}}/AN \rightarrow \mathbb{Z}/2$$

*defined by  $\Psi_\lambda(p \cdot AN) = e_\lambda(p) + 2\mathbb{Z}$  is a surjective group homomorphism. Furthermore,*

$$\Psi = \bigoplus_{[\lambda]} \Psi_\lambda: Q(H)^{\text{sym}}/AN \rightarrow \bigoplus_{[\lambda]} \mathbb{Z}/2,$$

*is an isomorphism, where  $[\lambda]$  runs over the equivalence classes of self-dual irreducible  $\lambda$ .*

*Proof of Theorem 1.1(1).* Choose a knot  $K_i$  for each  $i \in \mathbb{N}$  such that  $K_i$  are negative amphicheiral knots with irreducible Alexander polynomials  $\Delta_i(t) := \Delta_{K_i}(t)$  and the multisets  $C_i$  of nonzero coefficients of  $\Delta_i(t)$  are mutually distinct up to sign. We can find such knots, for instance using the knots in [Ch07, p. 60] whose Alexander polynomials are of the form  $a^2t^2 - (2a^2 + 1)t + a^2$ .

Let  $E_0$  be the exterior of the trivial (string) knot in  $D^2 \times [0, 1]$  and  $X = D^2 \times 0 \cap E_0$ . Let  $M = \Sigma_{g,n} \times [0, 1]$  and  $\iota: X \rightarrow \text{int}(\Sigma_{g,n})$  be an embedding which induces a nontrivial homomorphism on homology groups, and let  $f: E_0 \rightarrow \text{int}(M)$  be the embedding defined by  $f(x, t) = \iota(x, t/2 + 1/4)$ . Now for each  $K_i$  denote by  $E_{K_i}$  the exterior of  $K_i$  in  $S^3$  and define

$$M_i = (M - f(\text{int}(E_0))) \bigcup_{f(\partial E_0) = \partial E_{K_i}} E_{K_i}.$$

Since  $E_{K_i}$  and  $E_0$  have isomorphic homology groups,  $M_i = (M_i, \text{id}, \text{id})$  is a homology cylinder. By [CFK09, Proposition 4.3], the  $M_i$  generate an abelian subgroup of  $\mathcal{H}_{g,n}$  and we denote it by  $\mathcal{S}$ . Furthermore, if we denote by  $h$  the image of the generator  $H_1(E_0; \mathbb{Z}) \cong \mathbb{Z}$  under the homomorphism induced from  $f$ , then by [CFK09, Proposition 4.3]  $\tau(M_i) = \Delta_i(h)$ . Since  $h$  is an indivisible element in  $H = H_1(\Sigma_{g,n}; \mathbb{Z})$ , one can see that  $\Delta_i(h)$  is irreducible and self-dual. Moreover  $\Delta_i(h) \sim \Delta_j(h)$  if  $i \neq j$  since the multisets  $C_i$  are mutually distinct and invariants of  $\Delta_i(h)$  under the equivalence relation  $\sim$ . Therefore we deduce that

$$\Psi_{\Delta_i(h)}(\tau(M_j)) = \Psi_{\Delta_i(h)}(\Delta_j(h)) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise.} \end{cases}$$

This implies that the image of  $\Psi \circ \tau: \mathcal{H}_{g,n} \rightarrow \bigoplus_{[\lambda]} \mathbb{Z}/2$  is isomorphic with  $(\mathbb{Z}/2)^\infty$ . Moreover for an irreducible and self-dual element  $\lambda \in \mathbb{Z}[H]^\times$ , if  $\lambda \sim \Delta_i(h)$ , then  $\Psi_\lambda(\Delta_i(h)) = 0$ . Therefore we have a homomorphism  $\mathcal{H}_{g,n} \rightarrow (\mathbb{Z}/2)^\infty$  whose restriction to the abelian group  $\mathcal{S}$  is an isomorphism. Now the homomorphism splits and the abelian group  $\mathcal{S}$ , which is isomorphic to  $(\mathbb{Z}/2)^\infty$ , descends to a summand of the abelianization of  $\mathcal{H}_{g,n}$ .  $\square$

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DEPARTMENT OF MATHEMATICS AND POHANG MATHEMATICS INSTITUTE, POHANG UNIVERSITY OF SCIENCE AND TECHNOLOGY, POHANG GYUNGBUK 790–784, REPUBLIC OF KOREA

*E-mail address:* jccha@postech.ac.kr

MATHEMATISCHES INSTITUT, UNIVERSITÄT ZU KÖLN, GERMANY

*E-mail address:* sfriedl@gmail.com

DEPARTMENT OF MATHEMATICS, KONKUK UNIVERSITY, SEOUL 143–701, REPUBLIC OF KOREA

*E-mail address:* tkim@konkuk.ac.kr